L18 Locally most powerful tests with one-sided H_a

1. Concepts of locally most powerful (LMP) tests

   (1) Test classes

   $\psi(X)$ is a locally α-level test at $\theta_0 \overset{def}{\Longleftrightarrow} \theta_0 \in H_0$ and $\beta_\psi(\theta_0) \le \alpha$.
   $\phi(X)$ is an α-level test $\Longleftrightarrow \phi(X)$ is a locally α-level test at all $\theta_0 \in H_0$.

   The collection of all locally α-level tests at θ_0 is $\mathcal{L}_{\theta_0} = \{\psi(X) : \beta_\psi(\theta_0) \le \alpha\}$.
   Let $\mathcal{T}_{\theta_0} = \{\psi : \beta_\psi(\theta_0) \le \alpha$ and $[\beta_\psi(\theta)]'_\theta$ is continuous at $\theta_0\} \subset \mathcal{L}_{\theta_0}$ and consider selecting a good test in $\mathcal{T}_{\theta_0}$.

   (2) Locally most powerful test at θ_1 over all tests in $\mathcal{T}_{\theta_0}$

   $\phi(X)$ is a locally most powerful (LMP) test at θ_1 over $\mathcal{T}_{\theta_0}$
   $\overset{def}{\Longleftrightarrow}$ (i) $\exists\ \epsilon > 0$ such that $H_a \cap (\theta_1 - \epsilon,\ \theta_1 + \epsilon) \ne \emptyset$
   (ii) $\phi(X) \in \mathcal{T}_{\theta_0}$ and $\forall\ \psi(X) \in \mathcal{T}_{\theta_0}$ $\beta_\psi(\theta) \le \beta_\phi(\theta)$ $\forall\ \theta \in H_a \cap (\theta_1 - \epsilon,\ \theta_1 + \epsilon)$.

   So $\phi(X)$ is UMP test over $\mathcal{T}_{\theta_0} \Longleftrightarrow \phi(X)$ is LMP over $\mathcal{T}_{\theta_0}$ at all $\theta_1 \in H_a$.
   Comment: The test class is locally at θ_0 and the power comparison is locally at θ_1.

2. LMP test with upper-sided H_a

   Consider $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$ and $H_0 : \theta \le \theta_0$ versus $H_a : \theta > \theta_0$.

   (1) Definition

   $\phi(X)$ is LMP test at θ_0 over all tests in $\mathcal{T}_{\theta_0}$ if $\phi(X) \in \mathcal{T}_{\theta_0}$ and for all $\psi(X) \in \mathcal{T}_{\theta_0}$ there exists $\epsilon > 0$ such that $\beta_\psi(\theta) \le \beta_\phi(\theta)$ for all $\theta \in (\theta_0,\ \theta_0 + \epsilon)$.
   Comment: The test class and the power comparison are both locally at θ_0

   (2) Theorem

   $$\phi(X) = \begin{cases} 1 & f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) > 0 \\ r & f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) = 0 \\ 0 & f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) < 0 \end{cases} \quad \text{with } \int_x \phi(x) f(x;\,\theta_0)\,dx = \alpha. \text{ Then}$$

   (i) Assume $[\beta_\phi(\theta)]'_\theta$ is continuous. Then $\phi(X) \in \mathcal{T}_{\theta_0}$.
   (ii) If $\psi(X) \in \mathcal{T}_{\theta_0}$, excluding the cases where $\beta_\psi(\theta_0) = \beta_\phi(\theta_0)$ and $[\beta_\phi(\theta_0)]'_\theta = [\beta_\phi(\theta_0)]'_\theta$, then there exists $\epsilon > 0$ such that $\beta_\psi(\theta) \le \beta_\phi(\theta)$ for all $\theta \in (\theta_0,\ \theta_0 + \epsilon)$.

   So under the assumptions $\phi(X)$ is LMP test at θ_0 over $\mathcal{T}_{\theta_0}$.

   Proof. (i) is clearly true. (ii) For $\psi(X) \in \mathcal{T}_{\theta_0}$, $\int_x \psi(x) f(x;\,\theta_0)\,dx \le \alpha$.
   By generalized Neyman-Pearson lemma,

   $$[\beta_\psi(\theta_0)]'_\theta = \int_x \phi(x) f'_\theta(x;\,\theta_0)\,dx \le \int_x \phi(x) f'_\theta(x;\,\theta_0)\,dx = [\beta_\phi(\theta_0)]'_\theta.$$

   By the exclusion, $[\beta_\psi(\theta_0)]'_\theta < [\beta_\phi(\theta_0)]'_\theta$.
   Let $g(\theta) = \beta_\phi(\theta) - \beta_\psi(\theta)$. Then $g(\theta_0) \ge 0$ and $g'(\theta_0) > 0$. By the continuity of $g'(\theta)$, there exists $\epsilon > 0$ such that Let $g'(\xi) > 0$ on $\xi \in (\theta_0,\ \theta_0 + \epsilon)$. Thus on this interval

   $$g(\theta) = g(\theta_0) + g'(\xi)(\theta - \theta_0) \ge 0, \text{ i.e., } \beta_\psi(\theta) \le \beta_\phi(\theta).$$

3. LMP test with lower-sided H_a

Consider $H_0 : \theta = \theta_0$ versus $H_a : \theta < \theta_0$ and $H_0 : \theta \geq \theta_0$ versus $H_a : \theta < \theta_0$.

(1) Definition
$\phi(X)$ is LMP test at θ_0 over all tests in $\mathcal{T}_{\theta_0}$ if $\phi(X) \in \mathcal{T}_{\theta_0}$ and for all $\psi(X) \in \mathcal{T}_{\theta_0}$ there exists $\epsilon > 0$ such that $\beta_\psi(\theta) \leq \beta_\phi(\theta)$ for all $\theta \in (\theta_0 - \epsilon, \theta_0)$.
Comment: Test class and power comparison are both locally at θ_0.

(2) Theorem

$$\phi(X) = \begin{cases} 1 & -f'_\theta(x;\, \theta_0) - kf(x;\, \theta_0) > 0 \\ r & -f'_\theta(x;\, \theta_0) - kf(x;\, \theta_0) = 0 \\ 0 & -f'_\theta(x;\, \theta_0) - kf(x;\, \theta_0) < 0 \end{cases} \quad \text{with } \int_x \phi(x) f(x;\, \theta_0)\, dx = \alpha. \text{ Then}$$

(i) Assume $[\beta_\phi(\theta)]'_\theta$ is continuous. Then $\phi(X) \in \mathcal{T}_{\theta_0}$.
(ii) If $\psi(X) \in \mathcal{T}_{\theta_0}$, excluding the cases where $\beta_\psi(\theta_0) = \beta_\phi(\theta_0)$ and $[\beta_\phi(\theta_0)]'_\theta = [\beta_\phi(\theta_0)]'_\theta$, then there exists $\epsilon > 0$ such that $\beta_\psi(\theta) \leq \beta_\phi(\theta)$ for all $\theta \in (\theta_0 - \epsilon, \theta_0)$.

So under the assumptions $\phi(X)$ is LMP test at θ_0 over $\mathcal{T}_{\theta_0}$.

Proof. (i) is clearly true. (ii) For $\psi(X) \in \mathcal{T}_{\theta_0}$, $\int_x \psi(x) f(x;\, \theta_0)\, dx \leq \alpha$.
By generalized Neyman-Pearson lemma,

$$-[\beta_\psi(\theta_0)]'_\theta = \int_x \phi(x)[-f'_\theta(x;\, \theta_0)]\, dx \leq \int_x \phi(x)[-f'_\theta(x;\, \theta_0)]\, dx = -[\beta_\phi(\theta_0)]'_\theta.$$

By the exclusion, $[\beta_\psi(\theta_0)]'_\theta > [\beta_\phi(\theta_0)]'_\theta$.
Let $g(\theta) = \beta_\phi(\theta) - \beta_\psi(\theta)$. Then $g(\theta_0) \geq 0$ and $g'(\theta_0) < 0$. By the continuity of $g'(\theta)$, there exists $\epsilon > 0$ such that Let $g'(\xi) < 0$ on $\xi \in (\theta_0 - \epsilon, \theta_0)$. Thus on this interval

$$g(\theta) = g(\theta_0) + g'(\xi)(\theta - \theta_0) \geq 0, \text{ i.e., } \beta_\psi(\theta) \leq \beta_\phi(\theta).$$

Comment: If ϕ is UMP test for 2/3, then ϕ is LMP at θ_0. But constructing UMP monotone likelihood ratio in $T(X)$ is required.

Ex: With $\phi(X)$ in (2) of 3, suppose $\beta_\psi(\theta_0) \leq \alpha$. Show that if $[\beta_\psi(\theta_0)]'_\theta = [\beta_\phi(\theta_0)]'_\theta$, then $\beta_\psi(\theta_0) = \beta_\phi(\theta_0)$. Thus by exclusion, $[\beta_\psi(\theta_0)]'_\theta > [\beta_\phi(\theta_0)]'_\theta$.

Proof $\beta_\psi(\theta_0) \leq \alpha \iff \int_x \psi(x) f(x;\, \theta_0)\, dx \leq \alpha$. By Neyman-Pearson lemma

$$\int_x \psi(x)[-f'_\theta(x;\, \theta_0)]\, dx \leq \int_x \phi(x)[-f'_\theta(x;\, \theta_0)]\, dx$$

So $0 = \int_x (\phi - \psi)[-f'_\theta(x;\, \theta_0) - kf(x;\, \theta_0)]\, dx + \int_x (\phi - \psi) kf(x;\, \theta_0)\, dx$.
But $\int_x (\phi - \psi)[-f'_\theta(x;\, \theta_0) - kf(x;\, \theta_0)]\, dx \geq 0$ and $\int_x (\phi - \psi) kf(x;\, \theta_0)\, dx \geq 0$.
Thus $\int_x (\phi - \psi)[-f'_\theta(x;\, \theta_0) - kf(x;\, \theta_0)]\, dx = 0$ and $\int_x (\phi - \psi) kf(x;\, \theta_0)\, dx = 0$.
It follows $\int_x \psi(x) f(x;\, \theta_0)\, dx = \int_x \phi(x) f(x;\, \theta_0)\, dx$.

L19 Simplified LMP with one-sided H_a

1. Simple form of LMP with one-sided H_a

   Let $\mathcal{T}_{\theta_0} = \{\psi : \beta_\psi(\theta_0) \le \alpha$ and $[\beta_\psi(\theta)]'_\theta$ is continuous at $\theta_0\}$ and $U = \frac{f'_\theta(x;\,\theta_0)}{f(x;\,\theta_0)}$ where $f(x;\,\theta)$ is sample joint pdf/pmf.

   (1) For $H_0 : \theta = \theta_0$ vs $H_a : \theta > \theta_0$ and $H_0 : \theta \le \theta_0$ vs $H_a : \theta > \theta_0$,

   $$\begin{cases} 1 & f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) > 0 \\ r & f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) = 0 \\ 0 & f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) < 0 \end{cases} = \begin{cases} 1 & U > c \\ r & U = c \\ 0 & U < c \end{cases} = \phi(U)$$

   with $E_{\theta_0}[\phi(U)] = \alpha$, under certain assumptions, is LMP at θ_0 over $\mathcal{T}_{\theta_0}$.

   (2) For $H_0 : \theta = \theta_0$ vs $H_a : \theta < \theta_0$ and $H_0 : \theta \ge \theta_0$ vs $H_a : \theta < \theta_0$,

   $$\begin{cases} 1 & -f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) > 0 \\ r & -f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) = 0 \\ 0 & -f'_\theta(x;\,\theta_0) - kf(x;\,\theta_0) < 0 \end{cases} = \begin{cases} 1 & U < c \\ r & U = c \\ 0 & U > c \end{cases} = \phi(U)$$

   with $E_{\theta_0}[\phi(U)] = \alpha$, under certain assumptions, is LMP at θ_0 over $\mathcal{T}_{\theta_0}$.

   (3) Distribution of U: $U \sim AN\left(0,\, \frac{I(\theta_0)}{n}\right)$

   Proof. $U = \frac{f'_\theta(X_1,..,X_n;\,\theta_0)}{f(X_1,..,X_n;\,\theta_0)} = [\ln f(X_1,..,X_n;\,\theta_0)]'_\theta = [\ln f(X_1;\,\theta_0)]'_\theta + \cdots + [\ln f(X_n;\,\theta_0)]'_\theta$
   where $[\ln f(X_1;\,\theta_0)]'_\theta, ..., [\ln f(X_n;\,\theta_0)]'_\theta$ are iid $[\ln f(X;\,\theta_0)]'_\theta \sim (0,\, I(\theta_0))$.
   By CLT, $\frac{U}{n} \sim AN\left(0,\, \frac{I(\theta_0)}{n}\right)$. Hence $U \sim AN\left(0,\, nI(\theta_0)\right)$.

   Ex: Approximate the distribution of U with θ_0 by $N(0,\, nI(\theta_0))$.
   In (1) $\alpha = E_{\theta_0}[\phi(U)] \approx P(N(0,\, nI(\theta_0)) > c) = P\left(Z > \frac{c}{\sqrt{nI(\theta_0)}}\right)$
   $\Longrightarrow c = Z_\alpha\sqrt{nI(\theta_0)}$.
   In (2) $\alpha = E_{\theta_0}[\phi(U)] \approx P(N(0,\, nI(\theta_0)) < c) = P\left(Z < \frac{c}{\sqrt{nI(\theta_0)}}\right)$
   $\Longrightarrow c = -Z_\alpha\sqrt{nI(\theta_0)}$. Thus

$H_0 : \theta \le \theta_0$ vs $H_a : \theta > \theta_0$
Test statistic: $U = [\ln f(X_1,..,X_n;\,\theta_0)]'_\theta$
Reject H_0 if $U > Z_\alpha\sqrt{nI(\theta_0)}$

   is an approximate LMP test at θ over all tests in $\mathcal{T}_{\theta_0}$.

$H_0 : \theta \ge \theta_0$ vs $H_a : \theta < \theta_0$
Test statistic: $U = [\ln f(X_1,..,X_n;\,\theta_0)]'_\theta$
Reject H_0 if $U < -Z_\alpha\sqrt{nI(\theta_0)}$

   is an approximate LMP test at θ over all tests in $\mathcal{T}_{\theta_0}$.

2. Concept of LMP test with two-sided H_a

   Consider tests on $H_0 : \theta = \theta_0$ versus $H_a : \theta \ne \theta_0$.

(1) Test class

$$\mathcal{T} \;=\; \{\psi:\ \beta_\psi(\theta_0)\le\alpha \text{ and there exists } \delta>0 \text{ such that } \beta_\psi(\theta_0)\le\beta_\psi(\theta) \text{ for all } \theta\in(\theta_0-\delta_0)\cup(\theta_0,\,\theta_0+\delta)\}$$

is the collection of all locally α-level unbiased tests at θ_0.

$\mathcal{T}\cap\{\phi:\ [\beta_\psi(\theta)]''_{\theta^2}$ is continuous at $\theta_0\}\subset\mathcal{T}_2$ where

$$\mathcal{T}_2 \;=\; \left\{\psi:\ \beta_\psi(\theta_0)\le 0,\ \ [\beta_\psi(\theta_0)]'_\theta=0 \text{ and } [\beta_\psi(\theta)]''_{\theta^2} \text{ is continuous at } \theta_0\right\}$$

Comment: Test class $\mathcal{T}_2$ contains α-level unbiased tests locally at θ_0 with continuous second derivative of $\beta_\psi(\theta)$ at θ_0.

(2) Locally most powerful (LMP) test at θ_0 over all tests in $\mathcal{T}_2$

$\phi(X)$ is LMP test at θ_0 over $\mathcal{T}_2$ if $\phi(X)\in\mathcal{T}_2$ and for all $\psi(X)\in\mathcal{T}_2$ there exists $\epsilon>0$ such that $\beta_\psi(\theta)\le\beta_\phi(\theta)$ for all $\theta\in(\theta_0-\epsilon,\,\theta_0)\cup(\theta_0,\,\theta_0+\epsilon)$.

Comment: Power comparison for tests in $\mathcal{T}_2$ is locally at θ_0.

3. LMP α-level unbiased test

(1) Theorem

$H_0:\ \theta=\theta_0$ versus $H_a:\ \theta\ne\theta_0$. Let

$$\phi(X)=\begin{cases} 1 & f''_{\theta^2}(X;\,\theta_0)-k_1 f(X;\,\theta_0)-k_2 f'_\theta(X;\,\theta_0)>0\\ r & f''_{\theta^2}(X;\,\theta_0)-k_1 f(X;\,\theta_0)-k_2 f'_\theta(X;\,\theta_0)=0\\ 0 & f''_{\theta^2}(X;\,\theta_0)-k_1 f(X;\,\theta_0)-k_2 f'_\theta(X;\,\theta_0)<0\end{cases}$$

with $\int_x \phi(x)f(x;\,\theta_0)\,dx=\alpha$ and $\int_x \phi(x)f'_\theta(x;\,\theta_0)\,dx=0$. Then

(i) Assume that $[\beta_\phi(\theta)]''_{\theta^2}$ is continuous at θ_0. Then $\phi(X)\in\mathcal{T}_2$.

(ii) If $\psi(X)\in\mathcal{T}_2$ excluding the case where $\beta_\psi(\theta_0)=\beta_\phi(\theta_0)$, $[\beta_\psi(\theta_0)]'_\theta=[\beta_\phi(\theta_0)]'_\theta$ and $[\beta_\psi(\theta_0)]''_{\theta^2}=[\beta_\phi(\theta_0)]''_{\theta^2}$, then there exists $\epsilon>0$ such that $\beta_\psi(\theta)\le\beta_\phi(\theta)$ for all $\theta\in(\theta_0-\epsilon,\,\theta_0)\cup(\theta_0,\,\theta_0+\epsilon)$.

So under the assumptions $\phi(X)$ is LMP test at θ_0 over $\mathcal{T}_2$

(2) Proof

(i) is trivial.

(ii) For $\psi(X)\in\mathcal{T}_2$, $\int_x \psi(x)f(x;\,\theta_0)\,dx\le\alpha$ and $\int_x \psi(x)f'_\theta(x;\,\theta_0)\,dx=0$.

By generalized Neyman-Pearson lemma $\quad [\beta_\psi(\theta_0)]''_{\theta^2}\le[\beta_\phi(\theta_0)]''_{\theta^2}$.

By the exclusion $\quad [\beta_\psi(\theta_0)]''_{\theta^2}<[\beta_\phi(\theta_0)]''_{\theta^2}$.

Let $g(\theta)=\beta_\phi(\theta)-\beta_\psi(\beta)$. Then $g(\theta_0)\ge 0$, $g'(\theta_0)=0$ and $g''(\theta_0)>0$.

By the continuity, there exists $\epsilon>0$ such that $g''(\xi)>0$ on $(\theta_0-\epsilon,\,\theta_0)\cup(\theta_0,\,\theta_0+\epsilon)$.

On this interval

$$g(\theta)=g(\theta_0)+(\theta-\theta_0)g'(\theta_0)+\frac{1}{2!}(\theta-\theta_0)^2 g''(\xi)\ge 0$$

Hence $\beta_\psi(\theta)\le\beta_\phi(\theta)$ for all $\theta\in(\theta_0-\epsilon,\,\theta_0)\cup(\theta_0,\,\theta_0+\epsilon)$.